Lecture 3
Spin Dynamics

Lecture Notes by Assaf Tal

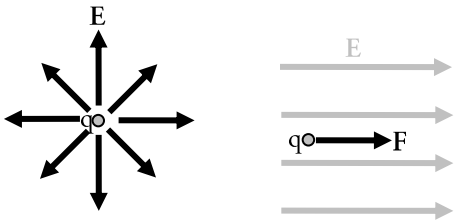
This lecture will begin by exploring the dynamics of single, uncoupled magnetic moments in a vacuum. We will explain how they interact with magnetic fields, and then expand our description to include macroscopic spin distributions.

The Magnetic Moment: The "Basic Unit" Of Magnetism

The Magnetic Dipole/Moment

Before talking about magnetic resonance, we need to recount a few basic facts about magnetism.

Electrodynamics is the field of study that deals with **magnetic fields** (**B**) and electric fields (**E**), and their interactions with matter. The basic entity that creates electric fields is the electric charge. For example, the electron has a charge, q, and it creates an electric field about it, $\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$, where **r** is a vector extending from the electron to the point of observation. The electric field, in turn, can act on another electron or charged particle by applying a force **F**=q**E**.



Left: a (stationary) electric charge q will create a radial electric field about it. Right: a charge q in an electric field will experience a force **F**=q**E**.

There is, however, no such thing as magnetic charge. The "elementary unit of magnetism" is the **magnetic moment**, also called the **magnetic dipole**.

[1] Magnetic fields are measured in Tesla (T) in the SI system of units. Other systems use the Gauss (G). The conversion is straightforward: 1 T = 10^4 G

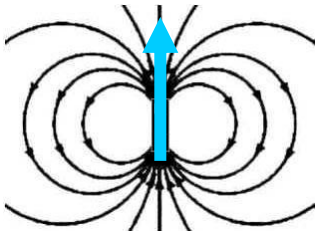
It is more complicated than charge because it is a vector, meaning it has both magnitude and direction. We will ask ourselves two basic questions:

1. What sort of magnetic fields does a magnetic moment create?
2. How does an external magnetic field affect the magnetic moment (apply force/torque, etc)?

We begin by answering the first question: Mathematically, if we have a magnetic moment **m** at the origin, and if **r** is a vector pointing from the origin to the point of observation, then it will give off a **dipolar field** described by:

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\frac{3(\boldsymbol{m}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{m}}{r^3}$$

The magnetic field can be visualized using a stream plot, which plots the field lines (to which **B** is parallel) at each point in space. These resemble in shape of an apple's core. These are sample field lines in the x-z plane (at y=0):

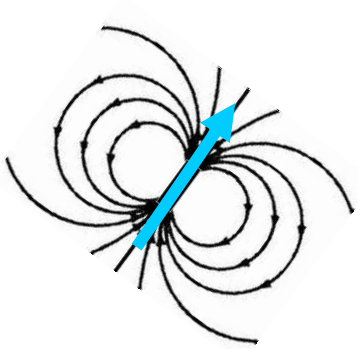


The field lines have cylindrical symmetry; i.e., they look the same through any plane which contains the z-axis at its center. The magnetic field is measured in units of **Tesla** (T) or **Gauss** (G) (1 T=10^4 G).

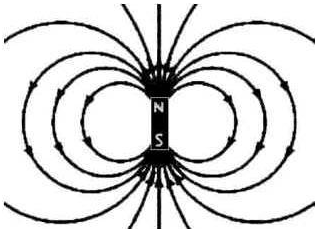
Number Time. The earth's magnetic field is about 0.5 G = $0.5\cdot10^{-4}$ T. Clinical MRI scanners operate at 1.5 T – 3.0 T, and the highest human MRI scanner as of early 2015 is the 11.75 Tesla human magnet being built in the University of Freiburg, Germany.

The magnitude of the generated magnetic field **B** is proportional to the size of the magnetic charge[1].

The direction of the magnetic moment determines the direction of the field lines. For example, if we tilt the moment, we tilt the lines with it:

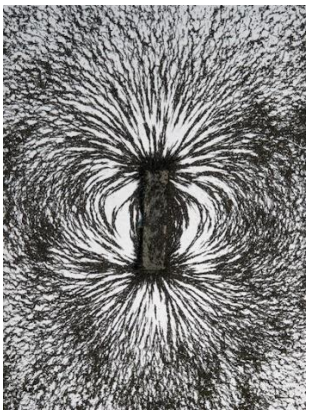


The simplest example of a magnetic moment is the refrigerator magnet. We'll soon meet other, much smaller and weaker magnetic moments, when we discuss the atomic nucleus.



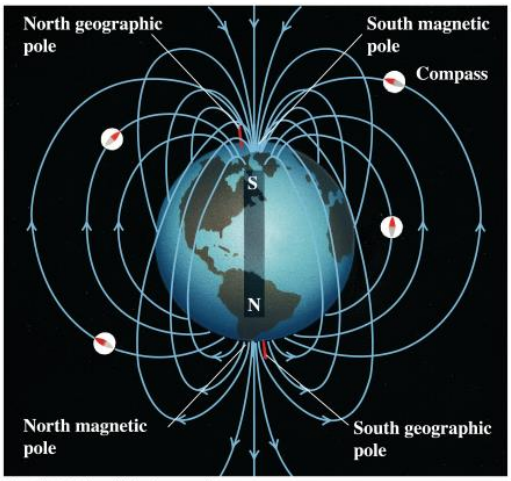
Your refrigerator magnet
has a permanent magnetic moment

Indeed, if you take such a piece of magnetized iron and place iron filings around it, the filings will align themselves along the field lines, illustrating them visually:



This also introduces the idea that "magnetic materials will align themselves" along the magnetic field they're in, but we're running ahead of ourselves here.

Another interesting example is the Earth itself, which behaves as if it had a giant magnetic moment stuck in its core, likely due to the presence of magnetized iron in its core:



Magnetic moments are measured in units of Joule/Tesla or (equivalently) in Ampere·meter^2 (1 J/T = 1 A·m^2).

Number Time. A typical refrigerator magnet might have a macroscopic magnetic moment of about 0.1 J/T. The tiny proton has an intrinsic magnetic moment equal to about $1.4 \cdot 10^{-26}$ J/T.

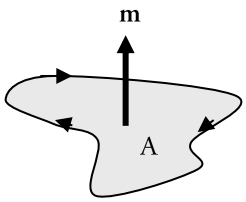
Magnetic Moments Are either Intrinsic or Induced

Magnetic moments are divided into two groups: current-induced and intrinsic.

Induced Moments: Basic electromagnetism tells us that a current flowing in a closed loop will give off a magnetic field. The loop can be *macroscopic*, like a wire, or *microscopic*, like an electron orbiting the nucleus. Far away from the current loop the field will look **as if** it were being generated by a magnetic dipole. If the magnetic loop is assumed to be planar, the magnetic dipole will be perpendicular to the loop, and have a magnitude given by

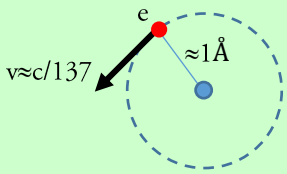
$$m=I\cdot A,$$

where I is the current in the loop and A is the area enclosed by the loop:



For a general (non-planar) current loop, the expression for **m** is somewhat more complicated, but the principle is the same.

Number Time. Classically, an electron orbis the nucleus at about the Bohr radius ($r \approx 1\text{Å} \approx 10^{-10}\text{m}$) with a velocity equal to about the fine structure constant times the speed of light: $v \approx c/137 \approx 2\cdot 10^6$ m/sec and angular frequency $\omega \approx \frac{2\pi v}{r}$.



Thus, its "effective current" is

$$I = \frac{\Delta C}{\Delta T} = \frac{1\ e}{2\pi/\omega} \approx 0.003\ \text{A}$$

The associated magnetic moment is:

$$m = AI = \pi r^2 I = 10^{-22}\text{A}\cdot\text{m}^2$$

Let's calculate the magnetic field created by this moment at the position of the electron itself:

$$|\boldsymbol{B}| \approx \left|\frac{\mu_0}{4\pi}\cdot\frac{3\boldsymbol{m}}{r^3}\right| \approx 30\ T.$$

This is an absolutely **enormous** magnetic field, much stronger than the ones employed in MRI! Although such effects are important in atomic physics, the orbital angular momentum of the electron is actually **zero** for most molecules and solids. This is due to an effect called **quenching of the orbital angular momentum**, where, due to reasons of symmetry in quantum mechanics – which we won't go into – the orbital angular momentum is forced to be zero (or almost zero).

Intrinsic Moments: It also appears that the fundamental particles – the proton, neutron and electron – carry **intrinsic** magnetic moments. That is, they "give off" a magnetic field as if a magnetic dipole were fixed to them, without having any current associated with them.

The angular momentum of elementary particles is measured in units of a fundamental constant known as Planck's constant (divided by 2π), $\hbar \approx 1.05\times 10^{-34}$ J·sec.

	Electron	Neutron	Proton
Charge (Coulombs)	-1.6×10^{-19}	0	1.6×10^{-19}
Mass (kg)	9.1×10^{-31}	1.6×10^{-27}	1.6×10^{-27}
Magnetic moment (J/T), $2\pi\gamma S$	9.26×10^{-24}	-0.96×10^{-26}	1.4×10^{-26}
Magnetic moment (μ_B)	-1.0	Irrelevant	Irrelevant
Magnetic moment (μ_N)	Irrelevant	-1.91	2.79
Spin, S (in units of $\hbar$)	1/2	1/2	1/2
Gyromagnetic ratio, γ (rad·Hz/T)	$2.8\cdot10^{10}$	-2.91×10^{7}	4.257×10^{7}

The **Bohr magneton**, μ_B, is just a quantity that makes it easy to talk about electron magnetism. It's not used often in nuclear magnetism, though:

$$\mu_B = \frac{e\hbar}{2m_e} = 9.27\times10^{-24}\ \frac{J}{T}$$

A similar quantity, the **nuclear magneton**, μ_N, is used more often in nuclear magnetism, although we won't be making direct use of it in these lecture notes:

$$\mu_N = \frac{e\hbar}{2m_p} = 5.05\times10^{-27}\ \frac{J}{T}$$

The phenomenon of intrinsic magnetic moments is directly related to another fundamental property of these particles called **spin**, and one speaks of a "nuclear spin" or an "electron spin". This is intrinsic angular momentum possessed by all electrons, protons and neutrons. Semi-classically, we can think of the proton or electron as a rotating ball of charge. The rotating charge can be thought of as loops of current, which give off a magnetic moment. In reality this picture is wrong, and you should always keep in mind spin is an intrinsic, somewhat weird quantum mechanical property; for example, the neutron has no charge and yet has a spin magnetic moment.

The semi-classical picture gets one thing right: the angular momentum and magnetic moment of the spinning sphere are parallel:

$$\mathbf{m}=\gamma\mathbf{S}.$$

The constant of proportionality is known as the **gyromagnetic ratio**, and is given in units of

$$[\gamma] = \frac{\text{Coulomb}}{\text{kg}} = \frac{\text{Hz}}{\text{Tesla}}$$

A word of caution about units: some books or tables quote γ in units of rad·MHz/T. For example, $\gamma=2\pi\cdot42.576$ rad·MHz/T for the hydrogen nucleus. Always be mindful of the units being used. Remember that, if we multiply γ by 2π, we will sometimes need to divide another quantity by 2π along the way. A simple example is that of the magnetic moment of the proton:

$$\boldsymbol{m} = \underbrace{\gamma}_{\substack{42.576\,\frac{\text{MHz}}{\text{T}}\\(\text{no }2\pi)}} \times \underbrace{\boldsymbol{S}}_{\substack{=\left(\frac{\hbar}{2}\right)\text{ for proton}\\(\text{has }2\pi)}}$$

Equivalently,

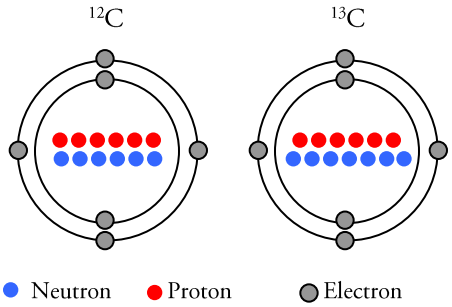
$$\boldsymbol{m} = \underbrace{\gamma}_{\substack{2\pi\cdot42.576\,\frac{\text{MHz}}{\text{T}}\\(\text{has }2\pi)}} \times \underbrace{\boldsymbol{S}}_{\substack{=\left(\frac{\hbar}{2}\right)\text{ for proton}\\(\text{no }2\pi)}}$$

In the second form, I moved the 2π factor from h to γ. The end result is the same, but now we must remember to specify the angular momentum in units without radians.

All electrons have an intrinsic magnetic moment, but that is not true for all nuclei, as we will see in the next section.

The Nuclear Magnetic Moment Is Determined by the Nucleus's Composition (Protons + Neutrons)

The nucleus is made up of protons and neutrons. The chemical name of an atom – carbon, hydrogen, phosphorous and so on – is determined by the number of protons it has. This will ultimately determine how many electrons it has and, therefore, its "chemistry". However, since neutrons are electrically neutral, their number might vary without changing the atom's "chemistry". Two such atoms are called **isotopes**. For example, shown here are two isotopes of carbon:



Two cartoon representations of ^{12}C (left), which has no nuclear spin, and ^{13}C (right), which has a nuclear spin of 1/2.

Proton and neutron spins tend to pair up anti-parallel due to the **Pauli exclusion principle**, in a manner similar to that of the electronic model of the atom, where levels fill up from lowest energy and up. This is quite surprising when you consider how strongly coupled the nucleons are, but it works. This reasoning works fairly well. For example, it predicts that **nuclei with an equal number of protons and neutrons should have 0 nuclear spin.** This works well for ^{12}C, ^{16}O, but not for ^{2}H, as shown by the next table:

Number of protons	Number of neutrons	Spin quantum number	Examples
Even	Even	0	^{12}C, ^{16}O, ^{32}S
Odd	Even	1/2	^{1}H, ^{19}F, ^{31}P
"	"	3/2	^{11}B, ^{35}Cl, ^{79}Br
Even	Odd	1/2	^{13}C
"	"	3/2	^{127}I
"	"	5/2	^{17}O
Odd	Odd	1	^{2}H, ^{14}N

It also predicts nuclei with an "extra" neutron or proton should have spin-½. This works for ^{13}C, ^{1}H, ^{31}P, ^{19}F, but not for ^{17}O. The breakdown of the pairing occurs before some nuclei have asymmetric nuclear charge distributions. These lead in some cases to favorable energy configurations with non-paired nucleons. Nuclei with spin>½ have asymmetric nuclear charge distribution and are known as **quadrupolar nuclei**, which interacts with the nuclear magnetic dipole and makes life very complicated. We won't discuss quadrupolar nuclei in this course.

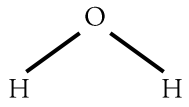
[2] Carbon has other isotopes but they do not occur naturally in nature and have zero natural abundance.

Nuclei with Low Natural Abundance Have "Low MRI Visibility"

It is very important to take into account the **natural abundance** of each isotope in determining how large its signal will be. The natural abundance tells us if we take N atoms of an element then, on average, what percentage of each isotope we will get.

Nuclei with low or very low natural abundance will be difficult to detect, simply because there are very few such nulei around. For example, ^{13}C has a natural abundance of about 1% and ^{12}C has a natural abundance of about[2] 99%. In a sample containing 100 carbon atoms, only about 1 will be a ^{13}C nucleus and the rest will be ^{12}C. Since only ^{13}C has a nuclear spin it will be the only one giving off a signal. In biological tissue, only about 1 in 100 carbons will give off a detectable MRI signal.

Natural abundance should also be kept in mind on the molecular level. Molecules are made out of atoms, connected between them by chemical bonds. The most important molecule in MRI is without a doubt water:



A "typical" water molecule actually comes in many isotopic flavors. Here are two examples:

Two isotopes of H_2O. The left is the most commonly found in nature. The one on the right is much rarer.

On the left is the most common variant by far. Oxygen-16 has no spin (its 8 protons pair up destructively, as do its 8 neutrons), and ^{1}H has spin ½. Because of symmetry, the two hydrogen atoms are equivalent, in the sense that they behave as **one** spin-1/2 entity with **double** the magnetic moment. The variant on the right is very rare, and has markedly different NMR properties (^{17}O has spin 5/2, and Deuterium has spin 1). Deviations from the "regular" H_2O are so rare, that their

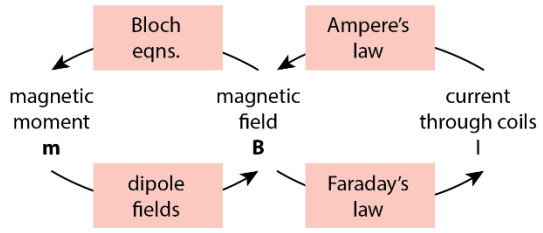
contribution to any experiment are negligible, as shown in the following table. Natural abundances are calculated by multiplying the natural abundances of the individual components (assuming statistical independence, which is an excellent assumption):

Oxygen	Hydrogen	Hydrogen	Nat. Ab. (%)
^{16}O	^{1}H	^{1}H	99.74
^{16}O	^{1}H	^{2}H	$9.97 \cdot 10^{-3}$
^{16}O	^{2}H	^{1}H	$9.97 \cdot 10^{-3}$
^{16}O	^{2}H	^{2}H	$9.98 \cdot 10^{-7}$
^{17}O	^{1}H	^{1}H	$3.99 \cdot 10^{-2}$
^{17}O	^{1}H	^{2}H	$3.99 \cdot 10^{-6}$
^{17}O	^{2}H	^{1}H	$3.99 \cdot 10^{-6}$
^{17}O	^{2}H	^{2}H	$4 \cdot 10^{-10}$
^{18}O	^{1}H	^{1}H	$1.99 \cdot 10^{-1}$
^{18}O	^{1}H	^{2}H	$1.99 \cdot 10^{-5}$
^{18}O	^{2}H	^{1}H	$1.99 \cdot 10^{-5}$
^{18}O	^{2}H	^{2}H	$2 \cdot 10^{-9}$

Thus, when we speak of **water** we're really neglecting all isotopic variants except for ^{16}O-^{1}H-^{1}H.

MRI Uses the Interaction of Magnetic Moments With Magnetic Fields

Just as electric charges give off electric fields and are affected by them, magnetic moments give off magnetic fields and are affected by them. This will turn out to be important since, as we'll see, we ourselves can create magnetic fields and pick them up using suitably constructed coils.



We've already noted that a moment will give off a magnetic (dipole) field. We therefore have three additional question we'd like to address in this lecture:

1. How do magnetic fields affect magnetic moments? The answer to that will come in the form of a set of equations known as the **Bloch Equations**, which will have a surprisingly simple solution.
2. How can we pick up magnetic fields using coils? Here, the answer will be by a process known as **induction**, by which time changing magnetic fields induce a voltage – and hence a current – in a coil of wire. The basic law of induction is known as **Faraday's law**.
3. How can we generate magnetic fields, thereby affecting the evolution of magnetic moments? The answer here will come in the form of **Ampere's Law**: current passed through a piece of wire or a coil will generate a magnetic field. The spatial distribution of the field will depend on the wire's shape, while its time characteristics will depend on the current as a function of time.

Magnetic Fields Cause Magnetic Moments to Precess: The Bloch Equations

How do magnetic fields affect magnetic moments? This is a question in basic electromagnetism, from which we will merely borrow the answer: as long as the wavelengths involved are long enough, which is the case for MRI, then:

1. **m** feels a force given by $\boldsymbol{F} = \boldsymbol{m} \cdot \nabla \boldsymbol{B}$.
2. **m** feels a torque given by $\boldsymbol{\tau} = \boldsymbol{m} \times \boldsymbol{B}$.

The force **F** turns out to be mostly negligible in-vivo, even for high magnetic fields of 3-10 T as found in human scanners. There are two reasons for this: First, the nuclear magnetic moment is very weak. Second, at the center of an MRI magnet, the magnetic field **B** is very uniform and varies by very little, so its spatial derivatives are small.

As for the torque, we have:

$$\frac{d\boldsymbol{m}}{dt} = \gamma \frac{d\boldsymbol{S}}{dt} = \gamma \boldsymbol{\tau} = \gamma \, \boldsymbol{m} \times \boldsymbol{B}$$

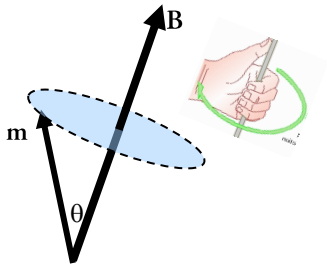
This equation is known as the **Bloch Equation** (BE). It is a vector equation and, therefore, actually three separate equations:

$$\begin{cases} m_x = \gamma (m_y B_z - m_z B_y) \\ m_y = \gamma \; m_z B_x - m_x B_z \\ m_z = \gamma (m_x B_y - m_y B_x) \end{cases}$$

These are three coupled first order linear differential equations. As far as differential equations they are considered very easy from a numerical point of view, but for a general time-dependent magnetic field they have no analytical solution. However, if the magnetic field is constant, their solution is quite straightforward, and I will quote here without proof. It is so important and fundamental that I'll put it in a textbox:

> A spin **m** in a time-constant magnetic field **B** will ***precess*** around the field **B** at an angular velocity $\omega=\gamma|\mathbf{B}|$ according to the left hand rule.

Curious readers are referred to the end of the chapter for a proof. Let's break this down slowly. First, a **precession** is a motion by which **m** traces out a cone around **B**, while keeping their angle θ fixed:



In precession, the tip of **m** traces out the dashed circle around **B**, while keeping θ fixed.

The sense of the rotation is determined using the left hand rule[3]: take your left hand and curl it with the thumb pointing along the field **B**. The way your fingers curl will tell you in which sense the magnetization is executing its precession. Finally, the angular velocity of the precession is fixed and given by $\omega=\gamma|\mathbf{B}|$ (a negative γ will reverse the sense of the rotation).

Since precession is really just a rotation of **m** about **B**, we can describe it mathematically using rotations. For example, if **B** is pointing along the z-axis, then **m** will simply rotate about the z-axis. A left-handed rotation matrix about z by an angle α is[4]:

$$R_z(\alpha)=\begin{pmatrix}\cos(\alpha) & \sin(\alpha) & 0\\ -\sin(\alpha) & \cos(\alpha) & 0\\ 0 & 0 & 1\end{pmatrix}$$

For a constant field, $\alpha=\omega t=\gamma B t$. If at time t=0 **m** points along the x-axis,

$$\boldsymbol{m}\ t=0\ \equiv \boldsymbol{m}_0=\begin{pmatrix}1\\0\\0\end{pmatrix},$$

then, for times $t\geq 0$,

$$\begin{aligned}\boldsymbol{m}\ t\ &= R\ \gamma B t\ \boldsymbol{m}_0\\ &=\begin{pmatrix}\cos\gamma Bt & \sin\gamma Bt & 0\\ -\sin\gamma Bt & \cos\gamma Bt & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}\\ &=\begin{pmatrix}\cos(\gamma Bt)\\ -\sin(\gamma Bt)\\ 0\end{pmatrix}\end{aligned}$$

Conceptually, any non-constant magnetic field **B**(t) can be broken down into very short time segments, δt. For short enough segments, **B** will be constant in each segment and we can predict its effect as a precession by some small amount around a fixed axis (which might change its orientation between time segments). Practically this might prove difficult for most cases, and will require a numerical solution.

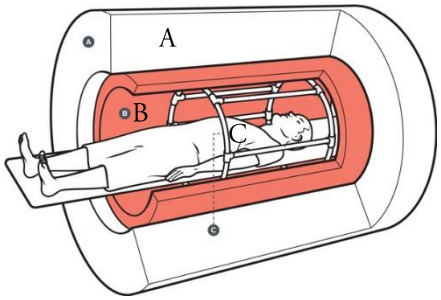
Spins Can Be Manipulated With Magnetic Fields: Ampere's Law

An MRI machine is basically just a collection of coils. We current is passed through a coil it generates a magnetic field, and it is through these magnetic fields that we control the nuclear magnetic

[3] A lot of practitioners make the mistake of using a right hand rule. This is incompatible with the Bloch equations, and if you think otherwise, please go ahead and solve them analytically for the simple case of a constant magnetic field **B** pointing along the z-axis and see for yourself.

[4] Often, you will forget where the minus sign should appear in a rotation matrix: on the sine term on the first or second row? Here's a quick trick: The **first column** of the matrix is the result of rotation a vector along x by an angle α. Set $\alpha=\frac{\pi}{2}$ for simplicity. For a left handed rotation, such a 90-degree rotation should take x into –y, or $(1,0,0)\rightarrow(0,-1,0)$. This holds if only if the minus sign appears on the sine term on the second row.

moments and produce an image. There are three major coil groups in the magnet:

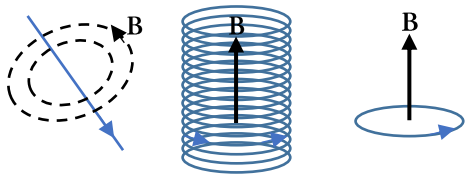


Main magnet coil (A), gradient coil (B) and body (RF) coil (C) inside a typical MRI scanner.

Ampere's law is also sometimes called Biot-Savart's law in some magnetism textbooks, but these refer to the same thing.

Ampere's law can be used to calculate the magnetic field generated by a current through a conductor. Three simple examples worth remembering are:

1. The magnetic field by a straight wire carrying currnet, which "goes around the wire" and decays off as $\sim\frac{1}{r}$, $\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\theta}}$.
2. The magnetic field inside an infinitely long solenoid carrying current I and having n turns per unit length: $\boldsymbol{B} = \mu n I\hat{\boldsymbol{z}}$, where z is the axis of the solenoid (independent of the radius and position in the solenoid).
3. The magnetic field along the axis of a loop of radius r carrying current I, at a height z relative to the plane of the loop (set at z=0): $\boldsymbol{B}(z) = \frac{\mu_0}{2}\frac{r^2 I}{(z^2+r^2)^{3/2}}\hat{\boldsymbol{z}}$. In particular, at the center of the loop: $\boldsymbol{B}(z) = \frac{\mu_0 I}{2r}\hat{\boldsymbol{z}}$.



Infinite wire Infinite solenoid Loop

Note that the magnetic field created is always proportional to the current. This is a natural consequence of the linearity of electromagnetism.

MRI-Generated Fields

The Main Field

A large cylindrical coil is wound along the patient's body. This coil is cooled with liquid helium and is superconducting, and can therefore carry large amounts of current without melting. Clinical scanners go up to 3 **Tesla**, which is about 60,000 times the Earth's magnetic field, which is 0.5 **Gauss** (1 T = 10^4 G). However, research scanners have already surpassed 10 T, although these are very expensive to build. The main field is usually called B_0 and its direction is taken to coincide with the z-axis:

$$\boldsymbol{B}_0 = \begin{pmatrix} 0 \\ 0 \\ B_0 \end{pmatrix}.$$

Number Time. For a clinical MRI scanner, B_0=3T. A proton nucleus (γ=2π·42.57 kHz/mT) will precess at a frequency of $\nu = \gamma B_0 / 2\pi = 127$ MHz, while a carbon nucleus (γ=2π·10.705 kHz/mT) will precess at about $\nu = \gamma B_0 / 2\pi = 32$ MHz about the main B_0 field. This precession frequency is called the **Larmor Frequency**.

The RF coils

The **radiofrequency (RF) coils** are capable of generating arbitrarily shaped, albeit weak (around 10 μT at most) field at the radiofrequency range. More precisely:

$$\boldsymbol{B}_{RF}\ t\ = \begin{pmatrix} B_{RF}\ t\ \cos(\phi_{RF}\ t\ ) \\ B_{RF}\ t\ \sin(\phi_{RF}\ t\ ) \\ 0 \end{pmatrix}.$$

We can shape the amplitude, $B_{RF}(t)$, and the phase, $\phi_{RF}(t)$, and create in theory any shape, although modern hardware limits our abilities somewhat (as noted earlier, peak $B_{RF}(t)$ is around 10 μT, and $\frac{d\phi_{RF}}{dt}$ ~radiofrequency range, usually tens or hundreds of MHz).

The Gradient Coils

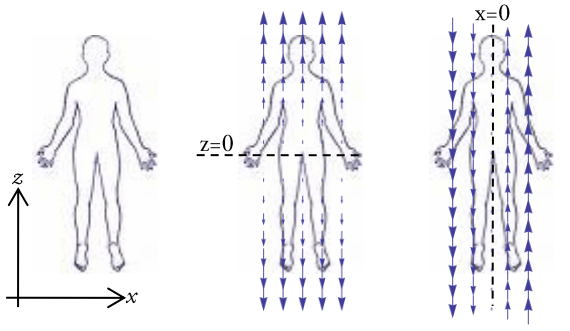
The **gradient coils** generate a linear, spatially varying magnetic field. So far, the RF and main fields have been spatially homogeneous, at least ideally. It is the gradient field that will enable us to image the sample. How precisely that will happen remains to be seen. For now, it suffices that we write down the general shape of the gradient field:

$$\boldsymbol{B}_{grad}\ \boldsymbol{r},t\ = \begin{pmatrix} 0 \\ 0 \\ \boldsymbol{G}\ t\ \cdot \boldsymbol{r} \end{pmatrix}.$$

Note we can "shape" the gradient field by shaping **G**(t), by shaping the current passing through the gradient coils. However, they are built to always be linear in position, **r**.

Number Time. The maximal gradient field strength is on the order of 10 mT/m, meaning over the human head (~ 0.2 m) one can create an additional z-field of about 10·0.2 mT ~ 1 mT.

It is important to understand visually what sort of fields the different gradient coils generate. The following illustration focuses on the case of a constant gradient:



No gradient
G=0
$\boldsymbol{B}_{grad} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

z-gradient
$\mathbf{G} = G\hat{\mathbf{z}}$
$\boldsymbol{B}_{grad} = \begin{pmatrix} 0 \\ 0 \\ G \cdot z \end{pmatrix}$

x-gradient
$\mathbf{G} = G\hat{\mathbf{x}}$
$\boldsymbol{B}_{grad} = \begin{pmatrix} 0 \\ 0 \\ G \cdot x \end{pmatrix}$

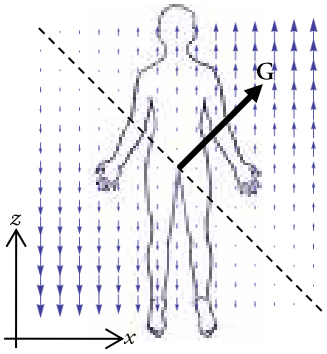
Effective field in the rotating frame for the cases of no gradient (left), z-gradient (middle) and x-gradient (right).

In all cases the gradient field superimposes a field pointing along the z-axis!

We can also turn on several gradient coils at once, generating a field which is a linear combination of the individual fields. For example, if we turn on both the x- and z-gradient fields at equal magnitude, the field will become

$$\boldsymbol{G} = \begin{pmatrix} G \\ 0 \\ G \end{pmatrix}, \quad \boldsymbol{B}_{eff} = \begin{pmatrix} 0 \\ 0 \\ G(x+z) \end{pmatrix}.$$

This is a linearly increasing field along an axis pointing along the direction of **G**:



Mixed gradient: **G**=(G,0,G)
$\boldsymbol{B}_{grad} = \begin{pmatrix} 0 \\ 0 \\ G \cdot (x+z) \end{pmatrix}$

Putting It All Together

The general, combined laboratory-generated magnetic field felt by a microscopic spin is therefore:

$$\begin{aligned} \boldsymbol{B}\ \boldsymbol{r},t\ &= \boldsymbol{B}_0 + \boldsymbol{B}_{RF}\ t\ + \boldsymbol{B}_{grad}\ \boldsymbol{r},t \\ &= \begin{pmatrix} B_{RF}\ t\ \cos(\phi_{RF}\ t\ ) \\ B_{RF}\ t\ \sin(\phi_{RF}\ t\ ) \\ B_0 + \boldsymbol{G}\ t\ \cdot \boldsymbol{r} \end{pmatrix} \end{aligned}$$

Microscopic Fields

The magnetic moments themselves create magnetic fields which affect each other. These will be treated in a short while.

MRI Happens In The Near Field

It is very important to keep in mind that almost all of the phenomena we will discuss in this course happen in the **near field**. This is a term used to describe distances that are small compared to the wavelengths involved. In general, any oscillating moment in free space with an angular frequency $\omega=2\pi\nu$ would create electromagnetic waves with a wavelength

$$\lambda = \frac{c}{\nu}$$

In a vacuum we have $c \approx 3\cdot 10^8 \frac{\text{m}}{\text{sec}}$, and for a hydrogen at 3T we have $\nu = \gamma B_0 \approx 127$ MHz, implying

$$\lambda \approx 2.4 \text{ m}.$$

Detection at distances << λ are said to be in the near field, which is precisely the case with MRI, in which the coils are placed as closely as possible to the subject.

The consequences of operating in the near field are subtle. For example, we've assumed a magnetic moment creates a dipolar magnetic field **B**(**r**) which changes immediately when we rotate the moment. This neglects the fact that field changes propagate at the speed of light (in a vacuum), which is permissible in the near field.

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\frac{3(\mathbf{m}(t)\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}(t)}{r^3}$$

The speed of light through a medium such as human tissue differs from that in vacuum, and is given by

$$v = \frac{c}{n} = \frac{c}{\sqrt{\varepsilon_r \mu_r}}$$

where c is the speed of light in vacuum, n the index of refraction, and ε_r, μ_r the (frequency dependent) relative permittivity and permeability of the medium. This makes wavelengths shorter and the near-field criterion more difficult to fulfill:

$$\lambda = \frac{c}{n\nu} = \frac{c}{\nu\sqrt{\varepsilon_r \mu_r}}.$$

For clinical field strengths (1.5 T and 3 T) this remains a reasonable-to-excellent approximation, depending on tissue type, but for ultra high field imaging (7 T and above) this assumption breaks down and correspondingly artifacts can be seen in the image. The table below shows some approximate values for these quantities at 1.5, 3 and 7 Tesla.

It should be noted that μ_r is not truly unity but very close, such that $\sqrt{\mu_r} = 1$ for all practical purposes. The true value of μ_r however cannot be neglected when calculating susceptibility artifacts (which we will not take upon ourselves in this course), since magnetic resonance is very sensitive to even small distortions in the main magnetic field.

Penetration Depth of RF Radiation

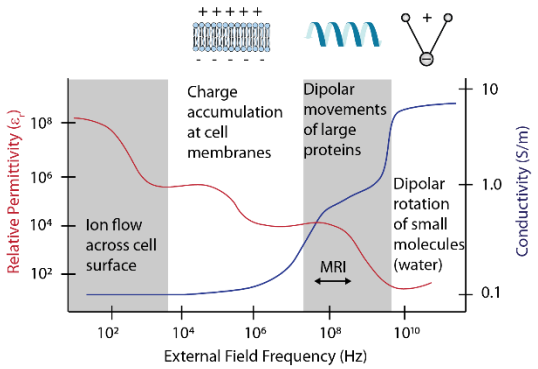
Another effect that must be taken into account is the conductance of the body's tissues, σ. It is quite amusing that conductivity, σ, is measured in units of Siemens/meter (in SI). The Siemens unit is called after the German scientist Ernst Werner Von Siemens, who also founded the Siemens company which today is a leading manufacturer of MRI scanners.

$$1 \text{ Siemens (S)} = \frac{1}{\text{Ohm}}$$

The electrical conductance tends to nonlinearly increase with increasing frequency, and so will be larger for the RF fields applied at 7T (300 MHz for protons) than 3T (127 MHz for protons). Typical

Material	Conductance σ (S/m)	Relative Permittivity ε_r	Relative Permeability μ_r	Field (T)	Wavelength λ (m)	Skin Depth δ (cm)
Vacuum	0	1	1	1.5	4.7	∞
	0	1	1	3	2.3	∞
	0	1	1	7	1.0	∞
Grey matter	0.33	97	1	1.5	0.48	11
	0.33	74	1	3	0.27	8
	0.33	60	1	7	0.13	5
White matter	0.33	68	1	1.5	0.57	11
	0.33	53	1	3	0.32	8
	0.33	44	1	7.0	0.15	5
Blood	0.15	86	1	1.5	0.51	16
	0.15	73	1	3	0.27	12
	0.15	65	1	7	0.12	8
Fat	0.04	6	1	1.5	1.92	31
	0.04	5.9	1	3	0.97	22
	0.04	5.6	1	7	0.43	15

conductance values for biological tissues range from 0.1-1.0 S/m. However, some materials, such as bone, conduct poorly ($\sigma < 0.1$), while the eyes conduct very well $\sigma \approx 1$. The rough dependence of conductivity and permittivity as a function of the frequency of the external field is shown below[5] (image courtesy of Dr. Rita Schmidt):



At low frequencies, ions flow freely without any resistance from the surrounding medium. As the frequency increases, some charge accumulation is observed on cell membranes. At even higher frequencies associated with MRI, the external fields induce random rotational motions in dipoles in large proteins and, then, in smaller water molecules, which provide a mechanism for conduction (effectively, dipoles can be thought of as tiny wire elements which can conduct current without actually moving as a whole).

An electromagnetic field with frequency ν will get mostly absorbed in any conductor with conductance σ (in ohms·meter) after traveling for a distance given by the **skin depth**[6]:

$$\delta = \sqrt{\frac{1}{\pi \nu \mu \sigma}} \approx \frac{503}{\sqrt{\mu_r \nu \sigma}} \text{ meters}$$

where μ_0 is the permittivity of the vacuum (Henry is an SI unit of inductance):

$$\mu_0 \approx 1.256 \times 10^{-6} \frac{\text{Henry}}{\text{meter}}$$

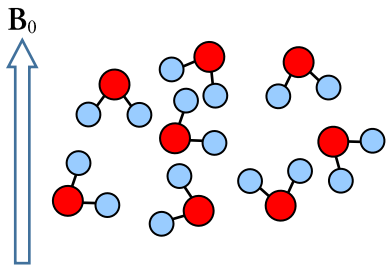
[5] Consult Prodan et. al., "The Dielectric Response of Spherical Live Cells in Suspension: An Analytic Solution", Biophys J (2008) for a discussion on the low-frequency response. Martinsen et. al. provide a more comprehensive review in the Encyclopedia of Surface and Colloid Science (2002), titled "Interface Phenomena and Dielectric Properties of Biological Tissue" (p. 2643). Finally, the book "RF / Microwave Interaction with Biological Tissues" by Vander Vorst et al (2006) also has in-depth discussions of the various physical mechanisms.

[6] This expression is valid for most materials far away from the so-called "plasma frequency", which is far above the radiofrequencies employed in MRI (which are in the MHz-GHz range).

This means that as we apply alternating magnetic fields to the human head, they will get attenuated substantially within a distance given approximately by the skin depth δ. This is because the alternating magnetic field will create electric fields which will move the free electrons inside the conductor and "do work" – namely, transfer energy to the electrons and as a result decay. The typical skin depth at 7T is only a few centimeters!

Spin Dynamics: What About Spin Ensembles?

In an MRI machine one cannot study single spins or single molecules, due to the low sensitivity of magnetic resonance. A typical voxel is ~ mm^3, and it often contains many many spins. MRI therefore studies the properties of nuclear spins *in bulk*. So far we've focused on the dynamics of a single spin, but what happens when we have an ensemble of spins? For example, if we have many water molecules – say, in a glass of water – and this glass is placed in an external, static $\mathbf{B}_0$ field?

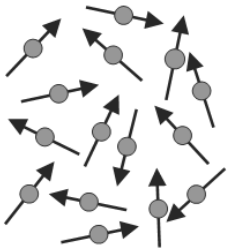


It is this problem that we address next. We will find out two main insights: First, that the external magnetic field polarizes the nuclear spins, and creates a net macroscopic nuclear magnetic moment; And, second, that the Brownian thermal rotational motion of the molecules leads to fluctuating magnetic fields which create relaxation – that is, return the system to thermal equilibrium if it is perturbed from it. This will lead us to modifying the Bloch equations to include relaxation in a phenomenological manner.

The Concept of Bulk Magnetization and Magnetic Moment

Suppose you have N molecules in a volume V, each having a magnetic moment $\mathbf{m}_i$. Recall that the moments are all vectors, so we can imaging a vector "attached" to each atom. In general, without the large external field of the MRI machine, they would all point in different directions:

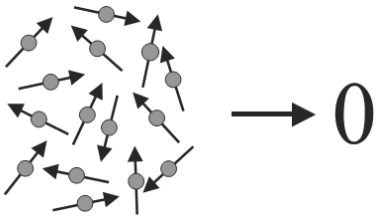
Schematic representation of an ensemble of microscopic magnetic moments. Each circle represents the magnetic moment of, say, a water molecule



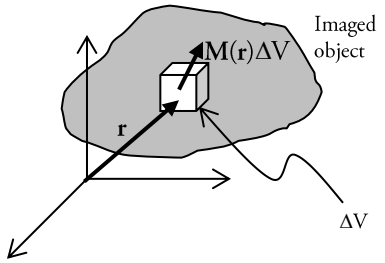
The **bulk magnetization** $\mathbf{M}$ of the volume V is defined as the (vector!) sum over all elements in the volume:

$$\mathbf{M}^{(\text{bulk})} = \sum_{i=1}^{N} \mathbf{m}_i$$

It is $\mathbf{M}^{(\text{bulk})}$ that MRI studies. In the above example, $\mathbf{M}^{(\text{bulk})}=0$ because the spins cancel out each other:



A related concept is that of **bulk magnetization per unit volume**, $\mathbf{M}(\mathbf{r})$, such that if we take a small volume ΔV around the point $\mathbf{r}$ then $\boldsymbol{M}\ \boldsymbol{r}\ \Delta V = \boldsymbol{M}^{\ bulk}$:



We will use the capital letters $\mathbf{M}$, $\mathbf{M}^{(\text{bulk})}$ to denote the macroscopic bulk magnetism properties, as

opposed to **m** which we will reserve for microscopic moments.

What volume ΔV should we use? On the one hand, we want enough spins in ΔV to make it statistically meaningful – that is, we want the variance of our fluctuations to be small as possible. Put another way, we want **M** to vary smoothly if we start shifting our volume of interest around.

Volume of Water	Number of spins
Liter = 10^3 cm^3	10^{25}
cm^3	10^{22}
mm^3	10^{19}
μm^3	10^{10}
$(10\ nm)^3$	10^4
nm^3	10

So it seems anything smaller than a box with sides 10 nm is shaky.

On the other hand, is there an upper limit on ΔV? A natural choice might be a voxel (~ mm^3), but there is a lot of variance inside a voxel. Sometimes to understand the signal originating from a single voxel we need to think in terms of what happens inside the voxel because there is a lot happening inside that mm^3. So, in general, we stick to the smallest ΔV we can take (say, $(10\ nm)^3$).

From now on when we talk about the magnetization vector we will take it to mean the bulk (macroscopic) magnetization vector per unit volume, unless specifically noted otherwise. At times I will remark how the macroscopic picture ties in with the microscopic one.

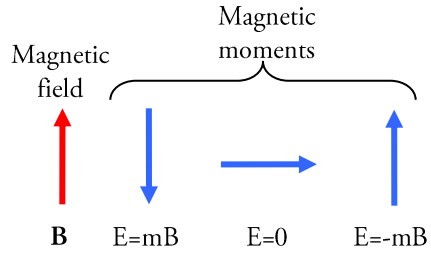
At Thermal Equilibrium, The Macroscopic Magnetic Moment is Parallel to the Main (B_0) Field

If you take a compass, which is nothing more than a magnetized iron needle, having a magnetic moment itself, it will align itself along the earth's magnetic field. This illustrates an important point of interest which we'll make use of: magnetic moments tend to align themselves along the magnetic field they are in when in equilibrium, in which they minimize the moment's energy:

$$E = -\boldsymbol{m} \cdot \boldsymbol{B} = -\mathrm{m} \cdot \mathrm{B} \cdot \cos\,\theta$$

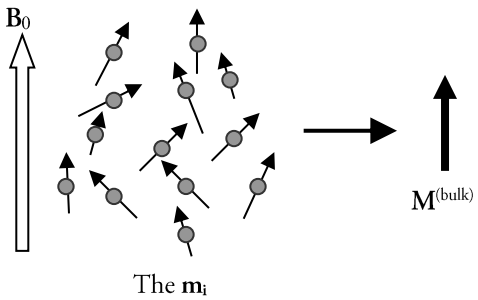
where θ is the angle between **m** and **B**. This phenomena is known as **paramagnetism**.

The energy E is at its minimum when **m** and **B** are parallel, and maximal when they are parallel:



A fundamental principle of statistical mechanics states that systems tend to minimize their energy, which explains why the compass needle aligns along **B**. However, one should be mindful that whether or not a macroscopic magnetic moment will actually align is dependent on competing interactions. For example, thermal motion might tend to randomize a magnetic moment's direction.
Question: why do microscopic spins precess about the magnetic field, instead of aligning along it?
Answer: Paramagnetism is an intrinsically ensemble phenomenon: a single spin cannot "align itself" along the external static magnetic field, because it has no mechanism by which to give away its energy (and therefore change θ). Instead, a single spin in vacuum would keep precessing around **B** forever. Only by coupling itself to the fluctuating time-dependent magnetic fields created by other spins around it can it relax over time.

Upon the application of an external field, the spins tend to align along the field – although thermal motion will prevent them from doing so completely. A "snapshot" of the spins in the presence of an external field might look like this:



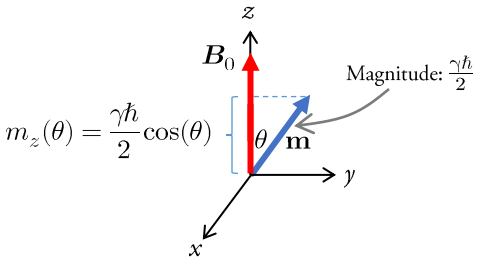
Calculating the Paramagnetic Equilibrium Bulk Magnetic Moment

We now come to the very important problem of calculating the bulk magnetic moment of a sample placed in a constant magnetic field (such as the 3 Tesla field of an MRI scanner) at thermal equilibrium. First, by symmetry, we expect that, on average, the microscopic magnetic moments will not have any orientation preference in the transverse (xy) plane, perpendicular to the main B_0 field: $\langle m_x \rangle = \langle m_y \rangle = 0$ at thermal equilibrium. This means we need to calculate $\langle m_z \rangle$. To this end, we will need a little bit of statistical physics and quantum mechanics. We will solve this for a spin-1/2 nuclei (^{1}H, ^{13}C, ^{31}P and most in-vivo relevant nuclei), and refer the reader to the literature for more complex derivations.

The z-component of the magnetization can assume several orientations, which we can parametrize by θ, the angle of **m** with the external B_0 field. Each such orientation has an energy associated with it:

$$E\ \theta\ = -\boldsymbol{m} \cdot \boldsymbol{B}_0 = -\frac{\gamma \hbar B_0}{2} \cos\ \theta$$

where $m = \frac{\gamma \hbar}{2}$ is the size of the microscopic nuclear magnetic moment.



$$m_z(\theta) = \frac{\gamma \hbar}{2} \cos(\theta)$$

Probability theory tells us that the average value of a quantity is given by the mean over all possible values, weighted by the probability of each value:

$$\langle m_z \rangle = \sum_{\theta} p\ \theta\ m_z\ \theta$$
$$= \sum_{\theta} p\ \theta\ \left(\frac{\gamma \hbar}{2}\right) \cos\ \theta$$

The summation extends over all possible angles θ of the magnetization (basically, from $-\pi$ to $+\pi$). At thermal equilibrium, the probability of the system being in a state with energy E is given by Bolztmann's distribution:

$$\Pr\ E\ = \frac{1}{Z} e^{-\frac{E}{kT}}$$

where Z a normalizing factor, called the partition function, given by:

$$Z = e^{-E_1/kT} + \ldots + e^{-E_n/kT}\ ,$$

where the system has N states having energies $E_1,\ldots,E_N$. The quantum mechanics we'll need says we only need to take into account what's known as the eigenstates of the system; in our case, only the parallel and antiparallel orientations of **m** need to be taken into account:

$$\theta = 0^\circ \quad m_\uparrow = \frac{\gamma \hbar}{2} \quad E_\uparrow = -\frac{\gamma \hbar B_0}{2}$$
$$\theta = 180^\circ \quad m_\downarrow = -\frac{\gamma \hbar}{2} \quad E_\downarrow = \frac{\gamma \hbar B_0}{2}$$

With this we have everything we need to calculate $\langle m_z \rangle$:

$$\langle m_z \rangle = p \uparrow m_\uparrow + p \downarrow m_\downarrow$$

First, we'll make a simple approximation: The energies associated with nuclear magnetism are much smaller than the thermal energy kT at room temperature:

$$\frac{\gamma \hbar B_0}{2} \approx 4 \times 10^{-26} J \quad \text{at } B_0 = 3T$$
$$kT \approx 4 \times 10^{-21} J \quad \text{at } T = 300K$$

which means we can approximate:

$$e^{-\frac{E}{kT}} \approx 1 - \frac{E}{kT}$$

As a result,

$$Z = e^{-\frac{E_\uparrow}{kT}} + e^{-\frac{E_\downarrow}{kT}} \approx 2$$
$$p \uparrow\ = \frac{1}{2}\left(1 - \frac{E_\uparrow}{kT}\right), p \downarrow\ = \frac{1}{2}\left(1 - \frac{E_\downarrow}{kT}\right)$$

We can plug these quantities back into our expression for $\langle m_z \rangle$, simplify and obtain:

$$\langle m_z \rangle = \frac{\gamma^2 \hbar^2 B_0}{4kT}$$
(at high temperatures)

Adding up N such nuclear spins we obtain the mean z-component of the bulk magnetization at thermal equilibrium, M_0:

$$M_0 \equiv N\langle m_z \rangle = \frac{N\gamma^2 \hbar^2 B_0}{4kT}$$
Equilibrium bulk macroscopic magnetic moment for spin–½ nucleus

Our assumption of non-interacting spins is a bit suspect, since the nuclear spins "talk" via dipolar coupling, but one can prove using quantum mechanics this holds even in the presence of dipolar and other interactions.

Looking at the above equation, we can write it as:

$$M_0 = N\left(\frac{\gamma\hbar}{2}\right)\left(\frac{\gamma\hbar B_0}{2kT}\right) = M_{\max}\cdot\left(\frac{\gamma\hbar B_0}{2kT}\right)$$

The quantity $M_{\max} \equiv N\left(\frac{\gamma\hbar}{2}\right)$ is the maximal moment you can get by aligning all of the spins completely along B_0. The factor $\left(\frac{\gamma\hbar B_0}{2kT}\right)$ represents the actual *fraction* of spins that are, on average (statistically) aligned along B_0.

Number Time. 1 mL of water will weigh about 1 gram. With a molecular weight of about 18 gr/mol, it has about $N=3\cdot10^{21}$ hydrogen atoms (and 2N proton nuclear spins). At room temperature ($kT\sim4\cdot10^{-21}$ $m^2\cdot kg\cdot s^{-1}$) and a field of B_0=3 Tesla, we have ($\gamma=2\pi\cdot42.576$ kHz/mT),

$$M_0 \approx 10^{-8}\ \text{J/T}.$$

Contrast this with the maximal moment:

$$M_{\max} \approx 10^{-3}\ \text{J/T}$$

which is about 5 orders of magnitude larger. This is a direct consequence of the fact that, due to thermal fluctuations and the smallness of the nuclear magnetic interaction, on average only about 1 spin in 100,000 align along the main field in a typical 3T MRI magnet! Mathematically,

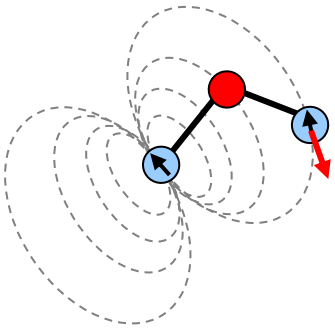
$$\frac{\gamma\hbar B_0}{2kT} \approx 10^{-5}$$

The above equation gives the macroscopic magnetic moment for N spins, but one could equally talk about magnetic moment per unit volume by dividing both sides by the volume, V. Then N would turn into the number density of spins, also known as the proton density when dealing with protons (number of nuclei per unit volume):

$$M_0 = \frac{PD\cdot\ \gamma\hbar\ ^2 B_0}{4kT}$$

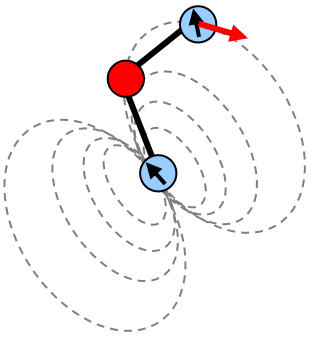
Spins Are Subjected To Microscopic Fluctuating Magnetic Fields Due To Their Thermal Motion, Which Induce Thermal Relaxation

Each microscopic nuclear magnetic moment **m** in our sample "sees" a magnetic field made up of two components: the macroscopic field generated by the coils in the lab, and the microscopic fields given off by its surroundings. For example, the dipolar field generated by one nuclear spin in a molecule will be felt by other nuclear spins in the same molecule.



Shown here is the magnetic field (red arrow) felt by one spin due to the dipolar field of the other spin in an H_2O molecule.

It's very important to realize that the orientation of the nuclear magnetic moment has nothing to do with the molecular orientation: if you rotate the molecule by 90°, the nuclear moment will not change, since it's not related to the nuclear charge or mass distribution; it "lives" in its own space and "talks" to the environment only through the magnetic fields it feels and emits:



Upon rotation of the molecule, the spins (black arrows) do not change their orientation. Consequently, the spin feels a different magnetic field, in both magnitude and direction.

Since most of the water molecules in the body are in the liquid state in the extra and intracellular matrices[7] –All molecules rotate and tumble around very rapidly. A small water molecule might perform a rotation on picosecond timescales, while larger molecules would rotate more slowly. This molecular rotation leads, by the arguments just laid out, to fluctuating intramolecular and intermolecular dipolar microscopic fields.

[7] This is actually not entirely correct, since water molecules often get "stuck" to cell membranes or confined in tight spaces. We will look more into this in later lectures.

The main effect of the fluctuating fields is to induce **thermal relaxation**: to bring the spins to a state of thermal equilibrium and maintain it. This is analogous to "friction" that dissipates energy in mechanical systems.

If the spins are perturbed from equilibrium – say, by the application of external magnetic fields – the microscopic fluctuations will work to bring them eventually back to thermal equilibrium.

Fluctuating Microscopic Fields Lead To Decoherence (T_2) And Return to Thermal Equilibrium (T_1)

The magnetic field felt by a microscopic nuclear magnetic moment can be subdivided into two parts, macroscopic and microscopic:

$$\boldsymbol{B}\ t\ = \boldsymbol{B}_{micro}\ t\ + \boldsymbol{B}_{macro}\ t$$

where the macroscopic fields are those generated by the laboratory coils and controlled by the scientist, and the microscopic fields are those fluctuating fields created by other spins in the molecule, electrons, and so forth. Consequently, the Bloch equations which describe the spin's precession become:

$$\frac{d\boldsymbol{m}}{dt} = \gamma \boldsymbol{m} \times \boldsymbol{B} = \gamma \boldsymbol{m} \times \boldsymbol{B}_{macro}\ t\ + \gamma \boldsymbol{m} \times \boldsymbol{B}_{micro}(t)$$

Now assume we have N magnetic moments, $\mathbf{m}_1$, $\mathbf{m}_2$, ..., $\mathbf{m}_N$, each experiencing its own unique microscopic field, but all experiencing the same macroscopic one:

$$\begin{aligned}
\frac{d\mathbf{m}_1}{dt} &= \gamma \mathbf{m}_1 \times \mathbf{B}_{macro}(t) + \gamma \mathbf{m}_1 \times \mathbf{B}^{(1)}_{micro}(t) \\
\frac{d\mathbf{m}_2}{dt} &= \gamma \mathbf{m}_2 \times \mathbf{B}_{macro}(t) + \gamma \mathbf{m}_2 \times \mathbf{B}^{(2)}_{micro}(t) \\
&\vdots \\
\frac{d\mathbf{m}_N}{dt} &= \gamma \mathbf{m}_N \times \mathbf{B}_{macro}(t) + \gamma \mathbf{m}_N \times \mathbf{B}^{(N)}_{micro}(t)
\end{aligned}$$

Tissue Type	Nuc.	Mol.	1.5 T		3 T		7 T	
			T_1	T_2	T_1	T_2	T_1	T_2
Gray Matter[b]	^1H	H_2O	1188 ± 69	95 ± 8	1820 ± 114	99 ± 7	2132 ± 103	
White Matter[b]	^1H	H_2O	656 ± 16	72 ± 4	1084 ± 45	69 ± 3	1220 ± 36	
Cerebrospinal Fluid[b]	^1H	H_2O	4070 ± 65				4425 ± 137	
Blood[b]	^1H	H_2O	1540 ± 23	290 ± 30	1932 ± 85	275 ± 50	2587 ± 283	
Kidney Cortex[a]	^1H	H_2O	966 ± 58	87 ± 4	1142 ± 154	76 ± 4		
Kidney Medula[a]	^1H	H_2O	1412 ± 58	85 ± 11	1545 ± 142	81 ± 8		
Liver[a]	^1H	H_2O	586 ± 39	46 ± 6	809 ± 71	34 ± 4		
Cartilage, 0° [d]	^1H	H_2O	1024 ± 70	30 ± 4	1168 ± 18	27 ± 3		
Cartilage, 55° [d]	^1H	H_2O	1038 ± 67	44 ± 5	1156 ± 10	43 ± 2		
Bone marrow (L4 vertebra)[a]	^1H	H_2O	549 ± 52	49 ± 8	586 ± 73	49 ± 4		
Prostate[a]	^1H	H_2O	1317 ± 85	88 ± 0	1597 ± 42	74 ± 4		
Subcutaneous fat[a]	^1H	Fat	343 ± 37	58 ± 4	382 ± 13	68 ± 4		
NAA CH_3 (GM)[c]	^1H	NAA	1270 ± 50		1470 ± 80	269 ± 7		
NAA CH_3 (WM)[c]	^1H	NAA	1360 ± 60		1400 ± 150	374 ± 9		

Typical T_1 and T_2 relaxation times from the literature, in milliseconds, in humans. The ± sign indicates standard deviation of the cohort examined. Note that variations may occur within a particular tissue (e.g. cortical vs. deep gray matter), and that numbers provided from different papers might originate from different regions within the same tissue. Also, some skepticism should be practiced when using values obtained for flowing/pulsating media, such as the cerebrospinal fluid.

[a] From: Bazelaire et. al., Radiology 230(3):652-659 (2004)

[b] T_1 values at 1.5 T and 7 T taken from Rooney et. al., Magn. Reson. Med. 57:308-318 (2007).
T_1, T_2 values at 3 T taken from Rooney et. al., Magn. Reson. Med. 57:308-318 (2007) and Stanisz et. al., Magn Reson Med 54:507-512 (2005).

[c] T_2 values at 3T taken from Kirov et. al., Magn. Reson. Med. 60:790-795 (2008).
T_1 values at 1.5T and 3T from Ethofer et. al., Magn Reson Med 50:1296-1301 (2003)

[d] From: Stanisz et. al., Magn Reson Med 54:507-512 (2005).

We now sum over multiple microscopic spins:

$$\sum_{n=1}^{N}\frac{d\mathbf{m}_n}{dt}=\gamma\sum_{n=1}^{N}\mathbf{m}_n\times\mathbf{B}_{macro}(t)+\gamma\sum_{n=1}^{N}\mathbf{m}_n\times\mathbf{B}_{micro}^{(n)}(t)$$

Since $\mathbf{B}_{macro}$ is common to all summed terms, and since the derivative of the sum equals the sum of the derivatives, we can substitute the microscopic moments by the macroscopic one, $\boldsymbol{M}^{bulk}=\sum_{n=1}^{N}\boldsymbol{m}_n$ and obtain:

$$\frac{d\boldsymbol{M}^{bulk}}{dt}=\gamma\boldsymbol{M}^{bulk}\times\boldsymbol{B}_{macro}\,t+\gamma\sum_{n=1}^{N}\boldsymbol{m}_n\times\boldsymbol{B}_{micro}^{n}\,t$$

The last term on the RHS represents the effects of the fluctuating fields and is intractable really. Physically speaking, these fluctuating magnetic fields are the source of (1) **decoherence** (i.e. loss of signal) and (2) **thermalization** (return to thermal equilibrium). Luckily, **phenomenologically** these effects can be respectively embodied by two constants, T_2 and T_1, respectively, which can be integrated into the Bloch equations using simple terms:

$$\begin{cases}M_x=\gamma(M_yB_z-M_zB_y)-\dfrac{M_x}{T_2}\\ M_y=\gamma\,M_zB_x-M_xB_z\,-\dfrac{M_y}{T_2}\\ M_z=\gamma(M_xB_y-M_yB_x)-\dfrac{M_z-M_0}{T_1}\end{cases}$$

We have omitted the subscripts and superscripts *macro* and *bulk*. Now the **B** that appears in our equations is the macroscopic field generated by the coils inside the MRI scanner and other macroscopic sources of field variation.

M_0 is the thermal equilibrium value of the magnetization, as can be seen by turning "off" the macroscopic RF and gradient fields, setting the time derivatives to 0 (which must be the case at equilibrium when macroscopic quantities do not change) and solving:

$$\begin{cases} M_x = -\dfrac{M_x}{T_2} \\ M_y = -\dfrac{M_y}{T_2} \\ M_z = -\dfrac{M_z - M_0}{T_1} \end{cases} \rightarrow \begin{cases} M_x = 0 \\ M_y = 0 \\ M_z = M_0 \end{cases}$$

A table of some T_1 and T_2 values has been compiled above. We note that for most in-vivo tissues at typical field strengths (1.5-3.0 Tesla), T_1 of protons in water is on the order of a second, while T_2 is on the order of 100 ms. Furthermore, T_1 tends to increase with increasing field strength, while T_2 tends to decrease. The field-dependence of T_1 and T_2 will await a further chapter which will discuss T_1 and T_2 as sources of contrast.

T_2 Leads To Decoherence

To gain a better understanding of the sort of effect T_2 has on the spins, let us set the macroscopic laboratory field to 0 and examine the time evolution of the magnetization.

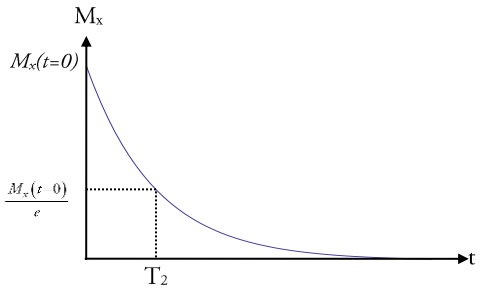
$$\begin{cases} M_x = -\dfrac{M_x}{T_2} \\ M_y = -\dfrac{M_y}{T_2} \\ M_z = -\dfrac{M_z - M_0}{T_1} \end{cases}$$

One interesting this is that the transverse (x, y) and longitudinal (z) components of the magnetization become decoupled: M_z does not feature in the equations for M_x and M_y, and M_x and M_y do not appear in the equation for M_z.

The equations for M_x and M_y have simple solutions:

$$M_x\ t\ = M_x\ t=0\ e^{-\frac{t}{T_2}}$$
$$M_y\ t\ = M_y\ t=0\ e^{-\frac{t}{T_2}}$$

This means that, whatever magnetization we start out with, it will decay with a time constant T_2 to zero:



This is called **decoherence**, and represents the physical fact that, unless something specific is done, the spins will point in all possible directions perpendicular to the MRI's static B_0 field, since there is no reason – energetic preference – for them to align in any single particular direction. The time T_2 can be thought of the time it takes M_x (or M_y) to drop to 1/e~37% of its initial value.

T_1 Leads To Thermal Equilibrium

At thermal equilibrium the spins align themselves along the external B_0 field. This is brought about by T_1 relaxation. The solution to the equation involving M_z is:

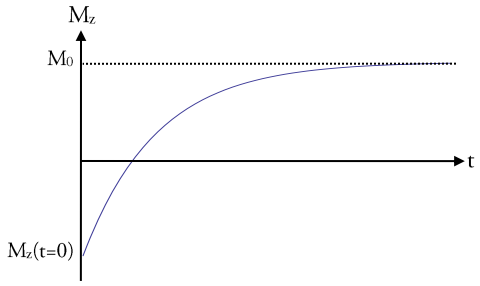
$$M_z\ t\ = M_z\ 0\ e^{-\frac{t}{T_1}} + \left(1 - e^{-\frac{t}{T_1}}\right) M_0$$

.

We see that, for t>>T_1,

$$M_z\left(t >> T_1\right) \approx M_0\,.$$

Thus, whatever longitudinal magnetization we start out from at t=0, it will converge back to its thermal equilibrium value M_0:



How the Magnetization Evolves in the Absence of Radiofrequency Fields: Precession + Thermal Relaxation

Next, we'd like to look at the slightly more general case in which the spins are subjected to the main (B_0) field and a potential constant field gradient, but not to a radiofrequency field (we'll deal with that later on). In this case,

$$\boldsymbol{B}_{macro}\ \boldsymbol{r}\ = \begin{pmatrix} 0 \\ 0 \\ B_0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \boldsymbol{G}\cdot\boldsymbol{r} \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ \Delta B(\boldsymbol{r}) \end{pmatrix}$$

The macroscopic Bloch equations become

$$\begin{aligned} \frac{dM_x}{dt} &= \gamma M_y \Delta B\ \boldsymbol{r}\ - \frac{M_x}{T_2} \\ \frac{dM_y}{dt} &= -\gamma M_x \Delta B\ \boldsymbol{r}\ - \frac{M_y}{T_2} \\ \frac{dM_z}{dt} &= -\frac{M_z - M_0}{T_1} \end{aligned}$$

We immediately see that the equation for the z-component, M_z, is the same. What changed were the equations for M_x and M_y, in two ways:

1. Each now has an additional term.
2. The term is dependent on position. This is not a problem, because we can fix **r** and just solve for each position **r** independently.
3. The equations for M_x and M_y are now mixed, making their solution slightly more complicated: the equation for M_x features M_y and vice versa.

There is a useful trick for solving both of these equations simultaneously, which rests on defining a new complex quantity, called the **transverse magnetization**:

$$M_{xy} \equiv M_x + iM_y$$

This is a complex number. We can think of it as a 2D "vector" in the real-imaginary plane: Its real component is given by M_x and its imaginary component by M_y.

On the one hand,

$$\frac{dM_x}{dt} + i\frac{dM_y}{dt} = \frac{d(M_x + iM_y)}{dt} = \frac{dM_{xy}}{dt}$$

On the other hand, we can use the x- and y-components of the Bloch equations to substitute for the left hand side of the above expression:

$$\begin{aligned} \frac{dM_x}{dt} + i\frac{dM_y}{dt} &= \left(\gamma M_y \Delta B\ \boldsymbol{r}\ - \frac{M_x}{T_2}\right) \\ &\quad + i\left(-\gamma M_x \Delta B\ \boldsymbol{r}\ - \frac{M_y}{T_2}\right) \\ &= -i\gamma \Delta B\ \boldsymbol{r}\ M_{xy} - \frac{M_{xy}}{T_2} \\ &= -\left(i\gamma \Delta B\ \boldsymbol{r}\ - \frac{1}{T_2}\right) M_{xy} \end{aligned}$$

This looks like a much simpler differential equation, of the form $\frac{dy}{dt} = -ay(t)$, for which the solution is $y\ t\ = y\ 0\ e^{-at}$. We solve by analogy:

$$M_{xy}\ t\ = M_{xy}\ 0\ e^{-i\gamma \Delta B\ \boldsymbol{r}\ t} e^{-\frac{t}{T_2}}$$

By taking the real and imaginary components of this equation, we can recover $M_x(t)$ and $M_y(t)$ (hint: recall $M_{xy}(0)$ is complex as well). However, it's often much easier to think about things in the complex plane:

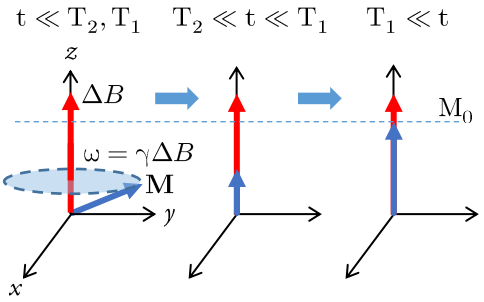
1. The factor $e^{-i\gamma \Delta B\ \boldsymbol{r}\ t}$ rotates the complex vector $M_{xy}(t)$ with an angular velocity $\omega\ \boldsymbol{r}\ = \gamma \Delta B(\boldsymbol{r})$.
2. The factor $e^{-\frac{t}{T_2}}$ "eats up" the size of $M_{xy}(t)$, causing it to diminish exponentially with a time constant T_2.

So, what we've shown is that, for a time-constant (but possible spatially heterogeneous) external magnetic field along the z-axis, the motion of the transverse magnetization is a "sum" of two motions: (1) Precession with angular frequency $\Delta B(\boldsymbol{r})$ + (2) Decay with time constant T_2.

Putting all of this together, we can say the following:

In the presence of a time-constant macroscopic magnetic field along the z-axis, the magnetization precesses about the z-axis and slowly decays with time constant T_2 in the xy plane and time constant T_1 along the z-axis.

For example, if we start at an angle θ to the z-axis and pointing along the x-axis, and if T_2 is shorter considerably than T_1 (as is often the case in-vivo):



Left: Initially, the spin precesses around the external field. **Middle**: As **M** precesses, T_2 "eats up" the transverse (xy) magnetization. T_1 has negligible effect on the effect at times $t<<T_1$. **Right**: Eventually, T_1 relaxation returns the z-component of **M** to its equilibrium value, M_0.

As a corollary, this example illustrates that:

The size of the macroscopic (bulk) magnetization is not conserved over time.

Before moving on, we tackle one last complication and allow the gradient fields to vary with time (as they can):

$$\boldsymbol{B}_{macro}\ \boldsymbol{r},t\ = \begin{pmatrix} 0 \\ 0 \\ B_0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \boldsymbol{G}(t)\cdot\boldsymbol{r} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \Delta B(\boldsymbol{r},t) \end{pmatrix}$$

We can repeat our method of solution from above step-by-step, defining the transverse magnetization and arriving at the same equation for M_{xy}, with the only difference being that $\Delta B\ \boldsymbol{r},t$ is now a function of time:

$$\frac{dM_{xy}}{dt} = -\left(i\gamma\Delta B\ \boldsymbol{r},t\ -\frac{1}{T_2}\right) M_{xy}$$

This cannot be directly integrated, for the same reason that the solution of $\frac{dy}{dt} = -a\ t\ y\ t$ is not $y\ t\ = y\ 0\ e^{-a\ t\ t}$. Instead, we must break up the time axis into N small chunks of length Δt, with total length $t = N\Delta t$, during each of which a(t) is approximately constant. Then the solution in each interval is

$$y\ t+\Delta t\ = e^{-a\ t\ \Delta t} y\ t\ .$$

The full solution is obtained by concatenating the short-time solutions:

$$\begin{aligned} y\ \Delta t\ &= y\ 0\ e^{-a\ 0\ \Delta t} \\ y\ 2\Delta t\ &= y\ \Delta t\ e^{-a\ \Delta t\ \Delta t} \\ &= y\ 0\ e^{-(a\ \Delta t\ +a\ 0\ )\Delta t} \\ y\ 3\Delta t\ &= y\ 2\Delta t\ e^{-a\ 2\Delta t\ \Delta t} \\ &= y\ 0\ e^{-(a\ 2\Delta t\ +a\ \Delta t\ +a\ 0\ )\Delta t} \end{aligned}$$

This can be continued by induction, with the sum turning into an integral as $\Delta t \to 0$:

$$y\ t\ = e^{-\int_0^t a\ t'\ dt'} y\ 0$$

We can apply the same reasoning when ΔB is time dependent and write:

$$M_{xy}\ t = M_{xy}\ 0\ \exp\left(-i\gamma\int_0^t \Delta B\ \boldsymbol{r},t'\ dt'\right)\exp\left(-\frac{t}{T_2}\right)$$

This equation describes the time evolution of the transverse magnetization under the action of a time-varying gradient in the absence of an external RF field. It is one of the most important equations in MRI, and one we will come back to many times throughout the course. Sometimes, we will write

$$\gamma\Delta B\ \boldsymbol{r},t\ = \Delta\omega\ \boldsymbol{r},t'$$

such that

$$M_{xy}\ t = M_{xy}\ 0\ exp\left(-i\int_0^t \Delta\omega\ \boldsymbol{r},t'\ dt'\right) exp\left(-\frac{t}{T_2}\right)$$

As mentioned above, our previous solution for M_z still holds:

$$M_z\ t\ = M_z\ 0\ e^{-\frac{t}{T_1}} + \left(1 - e^{-\frac{t}{T_1}}\right) M_0$$

B_0 Inhomogeneity Leads To Additional Transverse Decay (T_2')

An MRI magnet is built to yield a homogeneous field over a volume roughly the size of the human head. The ability of NMR and MRI to discern changes of ~Hz to the proton frequency means severe constraints are placed on the homogeneity. Indeed, a change of a single Hz would correspond to a change in the main field given by

$$\gamma\Delta B = 1\ \text{Hz}$$

or

$$\Delta B \approx 0.02\ \mu T.$$

This is

$$\frac{\Delta B}{B_0} \approx 6 \cdot 10^{-9}.$$

This is an incredibly difficult demand on the hardware: we need it to be homogeneous to about 0.01 ppm over a head-sized volume! The main coil's imperfections make it impossible to achieve. To approximate this requirement, special **passive shims** - pieces of iron - are added to the magnet to "shape" the main field.

Even if a perfect magnet is constructed, once we put in a sample, be it a human or an inanimate object, the main field will get distorted and its homogeneity would get ruined. Microscopically, human tissue is **diamagnetic**. This means an external field such as B_0 will induce magnetic moments in matter (of course the moments will induce a magnetic field which will create further moments which will induce further fields ... so the full solution must be self-consistent). The additional moment-induced field distorts the main field. The phenomenon is known as **magnetic susceptibility**. Susceptibility occurs on both a microscopic (cellular and sub-cellular) and macroscopic scales.

The effect of these inhomogeneities can be quite intricate, but they always lead to a decay of the signal. We'll delve into specific models later on, but for now we'll just state that on a small scale - on the order of a voxel or smaller - it can usually be modeled by swapping T_2 by a shorter time, T_2^*:

$$\frac{1}{T_2} \to \frac{1}{T_2^*}$$

In almost all realistic cases, T_2^* can be written down as a sum of two contributions: the microscopic decay and decay effects due to inhomogeneity:

$$\frac{1}{T_2} \to \frac{1}{T_2^*} = \underbrace{\frac{1}{T_2}}_{\substack{\text{microscopic, due to thermal motion of water molecules}}} + \underbrace{\frac{1}{T_2'}}_{\substack{\text{meso/microscopic, due to spatial variations in the magnetic field}}}$$